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Chern character for twisted K-theory of orbifolds

Jean-Louis Tu [a], Ping Xu [b,*,1]

[a] *Laboratoire de Mathématiques et Applications de Metz, Université de Metz, ISGMP, Bâtiment A, Ile du Saulcy, 57000 Metz, France*

[b] *Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA*

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Abstract

For an orbifold $\mathfrak{X}$ and $\alpha \in H^3(\mathfrak{X}, \mathbb{Z})$, we introduce the twisted cohomology $H_c^*(\mathfrak{X}, \alpha)$ and prove that the non-commutative Chern character of Connes–Karoubi establishes an isomorphism between the twisted K-groups $K_\alpha^*(\mathfrak{X}) \otimes \mathbb{C}$ and the twisted cohomology $H_c^*(\mathfrak{X}, \alpha)$. This theorem, on the one hand, generalizes a classical result of Baum–Connes, Brylinski–Nistor, and others, that if $\mathfrak{X}$ is an orbifold then the Chern character establishes an isomorphism between the K-groups of $\mathfrak{X}$ tensored with $\mathbb{C}$, and the compactly-supported cohomology of the inertia orbifold. On the other hand, it also generalizes a recent result of Adem–Ruan regarding the Chern character isomorphism of twisted orbifold K-theory when the orbifold is a global quotient by a finite group and the twist is a special torsion class, as well as Mathai–Stevenson's theorem regarding the Chern character isomorphism of twisted K-theory of a compact manifold.

Keywords: Twisted cohomology; Twisted K-theory; Orbifold; Gerbe; Chern character

Contents

* Corresponding author.
E-mail addresses: tu@univ-metz.fr (J.-L. Tu), ping@math.psu.edu (P. Xu).
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1. Introduction

Motivated by mathematical physics and especially string theory, there has been a great deal of interest in twisted K-theory [33]. A mathematically rigorous definition of the K-theory of a differentiable stack twisted by an S^1-gerbe was introduced in [32] and some fundamental properties were also established there. However, just as for the usual K-groups, it is in general a very difficult task to compute the twisted K-groups and very few examples are computed explicitly due to their complicated nature.

It is a classical result that for a compact manifold M, the Chern character establishes an isomorphism

$$K^*(M) \otimes \mathbb{C} \xrightarrow{\sim} H^*_{\mathrm{dR}}(M, \mathbb{C}).$$

Therefore, in a certain sense, modulo torsion, twisted K-groups are isomorphic to cohomology. It is therefore natural to ask what the twisted cohomology of a differentiable stack would be, so that the "Chern character" would give rise to an isomorphism. For a torsion class S^1-gerbe α over a compact manifold M, little changes and one can show that $K^*_\alpha(M) \otimes \mathbb{C}$ is isomorphic to $H^*_{\mathrm{dR}}(M, \mathbb{C})$. However, when the S^1-gerbe is of infinite order, some new phenomena appear, even in the manifold case. The usual way of defining the K_0-group as the Grothendieck group of vector bundles no longer works. Thus one must use an alternative definition of the "Chern character." Nevertheless, the twisted K-groups can be defined as the K-groups of some C^*-algebra (see [32]), and therefore one expects to use techniques of non-commutative differential geometry, and especially Connes–Karoubi's non-commutative Chern character map [10].

Recall that the non-commutative Chern character maps the K-groups of a smooth subalgebra (which is a dense topological algebra stable under the holomorphic functional calculus) of a C^*-algebra to its periodic cyclic homology. For a compact manifold M, Connes proved [10] that the periodic cyclic homology of $C^\infty(M)$ is isomorphic to the de Rham cohomology of M, and therefore the non-commutative Chern character map indeed generalizes the classical Chern character.

More precisely, for an S^1-gerbe α over a differentiable stack $\mathfrak{X}$, the twisted K-groups $K^*_\alpha(\mathfrak{X})$ are defined to be the K-groups of some C^*-algebra $C^*(\mathfrak{X}, \alpha) = C^*(\Gamma, L)$. It contains the subalgebra $C_c^\infty(\Gamma, L)$, which is stable under the holomorphic functional calculus when Γ is proper [32]. Here Γ is a Lie groupoid representing $\mathfrak{X}$ and $L = \widetilde{\Gamma} \times_{S^1} \mathbb{C}$ is the associated complex line bundle of the S^1-central extension $\widetilde{\Gamma} \to \Gamma$ representing the gerbe α. Thus an essential question is to study the periodic cyclic homology groups $HP_*(C_c^\infty(\Gamma, L))$.

In this paper, we confine ourselves to the case when $\mathfrak{X}$ is an orbifold, and thus Γ an étale proper groupoid. The main purpose of the paper is to study $HP_*(C_c^\infty(\Gamma, L))$. In this case, S^1-gerbes are classified by $H^3(\mathfrak{X}, \mathbb{Z})$ and therefore α can be considered as an element in $H^3(\mathfrak{X}, \mathbb{Z})$.

It is not surprising, as in the classical (i.e. non-twisted) case, that the inertia orbifold comes into the picture. This was shown by a classical theorem of Baum–Connes. In [3], Baum–Connes proved that when M is a manifold endowed with a proper action of a discrete group G, there is a Chern character isomorphism

$$\mathrm{ch}: K_i\big(C^*(M \rtimes G)\big) \to \bigoplus_{n\in\mathbb{N}} H_c^{i+2n}(\hat{M}/G, \mathbb{C}) \quad (i = 1, 2),$$

where $\hat{M} = \coprod_{g\in G} M^g \times \{g\}$. More generally, it is known that for any compact orbifold $\mathfrak{X}$ there is a Chern character isomorphism from $K_i(C^*(\mathfrak{X})) \otimes \mathbb{C}$ to the $\mathbb{Z}_2$-graded cohomology of the inertia orbifold $\bigoplus_{n\in\mathbb{Z}} H^{i+2n}(\Lambda\mathfrak{X}, \mathbb{C})$ ([1,8,11] in the global quotient case). See Section 2 below for the definition of inertia orbifolds. In the case when $\mathfrak{X} = M/G$, $\Lambda\mathfrak{X}$ is just $\hat{M}/G$.

In order to study the periodic cyclic homology groups $HP_*(C_c^\infty(\Gamma, L))$, we introduce the notion of twisted orbifold cohomology $H_c^*(\mathfrak{X}, \alpha)$. Let $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ be an S^1-gerbe over an orbifold $\mathfrak{X}$ with Dixmier–Douady class $\alpha \in H^3(\mathfrak{X}, \mathbb{Z})$. Let $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$ be an S^1-central extension representing this gerbe which admits a connection θ, curving B, and 3-curvature $\Omega \in \Omega^3(M)^\Gamma$. Denote by $L' \to S\Gamma$ the induced complex line bundle over the inertia groupoid $\Lambda\Gamma \rightrightarrows S\Gamma$, which is shown to admit a canonical flat connection. We denote by $\nabla': \Omega_c^*(S\Gamma, L') \to \Omega_c^{*+1}(S\Gamma, L')$ its corresponding covariant differential. We define the twisted cohomology groups (with compact supports) $H_c^*(\mathfrak{X}, \alpha)$ to be the cohomology of the complex

$$\big(\Omega_c^*(S\Gamma, L')^\Gamma((u)), \nabla' - 2\pi i \Omega u \wedge \cdot\big),$$

where u is a formal variable of degree -2, and $((u))$ are the formal Laurent series in u. When $\mathfrak{X}$ is a smooth manifold, this reduces to the twisted de Rham cohomology [23], which by definition is the cohomology of the complex $(\Omega^*(M)((u)), d - u\Omega \wedge \cdot)$. On the other hand, when α is a torsion class arising from a discrete torsion in the sense of [30], this reduces to the twisted orbifold cohomology of Ruan [30].

The main result of this paper can be outlined by the following

Theorem 1.1. *Let $\mathfrak{X}$ be an orbifold and $\alpha \in H^3(\mathfrak{X}, \mathbb{Z})$. Assume that $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$ is an S^1-central extension representing the S^1-gerbe determined by the class α, which admits a connection θ, a curving B, and a 3-curvature Ω. Then there are isomorphisms*

$$K_\alpha^*(\mathfrak{X}) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}} HP_*\big(C_c^\infty(\Gamma, L)\big) \to H_c^*(\mathfrak{X}, \alpha),$$

where ch *denotes the non-commutative Chern character of Connes–Karoubi.*

Our theorem generalizes, on the one hand, the recent theorem of Mathai–Stevenson [23] concerning the non-commutative Chern character for twisted K-theory of a compact manifold, and on the other hand, a theorem of Adem–Ruan [1] regarding the Chern character of twisted K-theory of orbifolds when the orbifold is a global quotient by a finite group G and the S^1-gerbe is a torsion class induced from a central extension of G.

Note that the periodic cyclic homology of (untwisted) groupoid algebras has been studied extensively by many authors, including Burghelea [9] in the case of discrete groups, Feigin–Tsygan [16] and Nistor [26] in the case of a discrete group acting on a manifold, Baum–Brylinski–MacPherson [2] and Block–Getzler [6] in the case of a compact Lie group acting on a compact manifold, and Brylinski–Nistor [8] and Crainic [11] in the case of étale groupoids.

The paper is organized as follows. Section 2 recalls some basic materials concerning S^1-gerbes over orbifolds including the definition of twisted K-theory of orbifolds. Section 3 introduces twisted cohomology. Section 4 is devoted to the proof of the main theorem by introducing a natural chain map between the chain complex of the periodic cyclic homology and that of twisted cohomology.

Note that, besides orbifolds, another important case of differentiable stacks would be quotient stacks, namely those corresponding to transformation groupoids. Twisted cohomology and the Chern character for this case will be discussed in a separate paper.

2. S^1-gerbes over orbifolds

In this section we recall a few basic facts concerning orbifolds and S^1-gerbes over orbifolds.

2.1. *Orbifolds*

Roughly speaking, an orbifold is obtained by gluing charts consisting of manifolds endowed with an action of a finite group. To an orbifold $\mathfrak{X}$, one can associate an étale proper Lie groupoid $\Gamma \rightrightarrows M$, called a presentation of the orbifold. Note that presentations of an orbifold are not unique. However, they are uniquely determined up to Morita equivalence [24,32]. In other words, there is a one-to-one correspondence between Morita equivalence classes of étale proper groupoids and orbifolds. The topological space $|\mathfrak{X}|$ underlying the orbifold $\mathfrak{X}$ is then the orbit space $|\Gamma| := M/\Gamma$. Recall that a groupoid $\Gamma \rightrightarrows M$ is étale if both the target and source maps $t, s : \Gamma \to M$ are local diffeomorphisms, and it is proper if the map $(t, s) : \Gamma \to M \times M$ is proper.

By a *refinement* of an étale proper groupoid $\Gamma \rightrightarrows M$, we mean a pair $(\Gamma' \rightrightarrows M', f)$, where $\Gamma' \rightrightarrows M'$ is an étale proper groupoid and $f : \Gamma' \to \Gamma$ is an étale groupoid morphism which induces a Morita equivalence. That is to say, (i) $f_0 : M' \to M$ is an étale map which induces a surjection $|\Gamma'| \to |\Gamma|$, and (ii) the diagram

$$\begin{array}{ccc} \Gamma' & \xrightarrow{s'\times t'} & M' \times M' \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{s\times t} & M \times M \end{array}$$

is Cartesian. For instance, if (U_i) are open subsets of M such that (U_i/Γ) is a cover of $|\Gamma|$, let $\Gamma' = \coprod_{i,j} \Gamma_{U_j}^{U_i}$, where

$$\Gamma_{U_j}^{U_i} = \{ g \in \Gamma \mid s(g) \in U_j \text{ and } t(g) \in U_i \}.$$

Then Γ' is endowed with a groupoid structure with unit space $M' = \coprod_i U_i$, product $(i,j,g)(j,k,h) = (i,k,gh)$ and inverse $(i,j,g)^{-1} = (j,i,g^{-1})$. Then $\Gamma' \rightrightarrows M'$, together with the map $f(i,j,g) = g$, is a refinement of Γ.

An étale proper groupoid $\Gamma \rightrightarrows M$ is said to be *nice* if for all n, Γ_n is a disjoint union of contractible open subsets. The following result is proved in [25, Corollary 1.2.5].

Proposition 2.1. *Any étale proper groupoid admits a nice refinement.*

2.2. S^1-gerbes over orbifolds

An S^1-gerbe over an orbifold $\mathfrak{X}$ is represented by an S^1-central extension $S^1 \to \widetilde{\Gamma} \to \Gamma \rightrightarrows M$ of a groupoid $\Gamma \rightrightarrows M$ representing $\mathfrak{X}$. More precisely, if $\Gamma \rightrightarrows M$ is an étale proper Lie groupoid representing an orbifold $\mathfrak{X}$, there is a one-to-one correspondence between S^1-central extensions of $\Gamma \rightrightarrows M$ and S^1-gerbes $\widetilde{\mathfrak{X}}$ over $\mathfrak{X}$ whose restriction to M: $\widetilde{\mathfrak{X}}|_M$ admits a trivialization [4,5]. We refer the reader to [4,5] for the general theory of the relation between S^1-central extensions of Lie groupoids and S^1-gerbes over differentiable stacks.

Below we recall some basic definitions which are needed in this paper.

Definition 2.2. Let $\Gamma \rightrightarrows M$ be a Lie groupoid. An S^1-*central extension* of $\Gamma \rightrightarrows M$ consists of

(1) a Lie groupoid $\widetilde{\Gamma} \rightrightarrows M$, together with a morphism of Lie groupoids $(\pi, \mathrm{id}) : [\widetilde{\Gamma} \rightrightarrows M] \to [\Gamma \rightrightarrows M]$, and
(2) a left S^1-action on $\widetilde{\Gamma}$, making $\pi : \widetilde{\Gamma} \to \Gamma$ a (left) principal S^1-bundle.

These two structures are compatible in the sense that $(s \cdot x)(t \cdot y) = st \cdot (xy)$, for all $s, t \in S^1$ and $(x, y) \in \widetilde{\Gamma} \times_M \widetilde{\Gamma}$.

The following equivalent definition is quite obvious, which indicates that this is indeed a generalization of usual group S^1-central extensions.

Proposition 2.3. *Let $\Gamma \rightrightarrows M$ be a Lie groupoid. A Lie groupoid $\widetilde{\Gamma} \rightrightarrows M$ is an S^1-central extension of $\Gamma \rightrightarrows M$ if and only if there is a groupoid morphism $\pi : \widetilde{\Gamma} \to \Gamma$, which is the identity when being restricted to the unit spaces M such that its kernel* $\ker \pi$ *is isomorphic to the bundle of groups $M \times S^1$ and lies in the center of $\widetilde{\Gamma}$.*

We now recall the definition of Morita equivalence of S^1-central extensions [5,32].

Definition 2.4. We say that two S^1-central extensions $S^1 \to \widetilde{\Gamma} \to \Gamma \rightrightarrows M$ and $S^1 \to \widetilde{\Gamma}' \to \Gamma' \rightrightarrows M'$ are Morita equivalent if there exists an S^1-equivariant $\widetilde{\Gamma}_\bullet$–$\widetilde{\Gamma}'_\bullet$-bitorsor Z, by which we mean that Z is a $\widetilde{\Gamma}_\bullet$–$\widetilde{\Gamma}'_\bullet$-bitorsor endowed with an S^1-action such that

$$(\lambda r) \cdot z \cdot r' = r \cdot (\lambda z) \cdot r' = r \cdot z \cdot (\lambda r')$$

whenever $(\lambda, r, r', z) \in S^1 \times \widetilde{\Gamma} \times \widetilde{\Gamma}' \times Z$ and the products make sense.

In the sequel, we will identify an S^1-gerbe over an orbifold $\mathfrak{X}$ with the Morita equivalence class of S^1-central extensions of Lie groupoids representing the gerbe.

Denote by $\mathcal{S}^1$ (respectively $\mathcal{R}$) the sheaf of S^1-valued (respectively $\mathbb{R}$-valued) smooth functions. The following is a well-known theorem of Giraud [17].

Theorem 2.5 *(Giraud). Isomorphism classes of S^1-gerbes over $\mathfrak{X}$ are in one-to-one correspondence with $H^2(\mathfrak{X}, \mathcal{S}^1)$.*

The exponential sequence $0 \to \mathbb{Z} \to \mathcal{R} \to \mathcal{S}^1 \to 0$ gives rise to a long exact sequence:

$$\cdots \to H^2(\mathfrak{X}, \mathcal{R}) \to H^2(\mathfrak{X}, \mathcal{S}^1) \xrightarrow{\phi} H^3(\mathfrak{X}, \mathbb{Z}) \to H^3(\mathfrak{X}, \mathcal{R}) \to \cdots. \tag{1}$$

Given an S^1-gerbe $\widetilde{\mathfrak{X}}$ over $\mathfrak{X}$, we call the image $\phi([\widetilde{\mathfrak{X}}]) \in H^3(\mathfrak{X}, \mathbb{Z})$, its *Dixmier–Douady* class, where $[\widetilde{\mathfrak{X}}] \in H^2(\mathfrak{X}, \mathcal{S}^1)$ its isomorphism class as in Theorem 2.5. Since an orbifold can be represented by an étale proper groupoid, we have $H^2(\mathfrak{X}, \mathcal{R}) = 0$ and $H^3(\mathfrak{X}, \mathcal{R}) = 0$, and therefore

$$H^2(\mathfrak{X}, \mathcal{S}^1) \xrightarrow{\phi} H^3(\mathfrak{X}, \mathbb{Z})$$

is indeed an isomorphism [32, Proposition 2.22]. Thus we have

Theorem 2.6. *Isomorphism classes of S^1-gerbes over an orbifold $\mathfrak{X}$ are classified by $H^3(\mathfrak{X}, \mathbb{Z})$.*

Thus for an orbifold, Dixmier–Douady classes completely classify S^1-gerbes over it, which is in general false for a differentiable stack [32]. Given an S^1-central extension $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$ as above, let $L = \widetilde{\Gamma} \times_{S^1} \mathbb{C}$ be its associated complex line bundle. Then $L \to \Gamma \rightrightarrows M$ is equipped with an associative bilinear product

$$\begin{aligned} L_g \otimes L_h &\to L_{gh}, \quad \forall (g,h) \in \Gamma_2, \\ (\xi, \eta) &\mapsto \xi \cdot \eta, \end{aligned}$$

and an antilinear involution

$$\begin{aligned} L_g &\to L_{g^{-1}}, \\ \xi &\mapsto \xi^*, \end{aligned}$$

satisfying the following properties:

- the restriction of the line bundle to the unit space M is isomorphic to the trivial bundle $M \times \mathbb{C} \to M$;
- $\forall \xi, \eta \in L_g$, $\langle \xi, \eta \rangle = \xi^* \cdot \eta \in L_{s(g)} \cong \mathbb{C}$ defines a scalar product;
- $(\xi \cdot \eta)^* = \eta^* \cdot \xi^*$.

Example 2.7. Note that when Γ is a nice étale proper groupoid, then all S^1-central extensions are topologically trivial. Therefore they are in one–one correspondence with S^1-valued 2-cocycles of the groupoid Γ, i.e. differentiable maps $c : \Gamma_2 \to S^1$ satisfying the relation

$$c(g,h)c(gh,k) = c(h,k)c(g,hk), \quad \forall (g,h,k) \in \Gamma_3,$$

modulo coboundaries, i.e. cocycles of the form $c(g,h) = b(g)b(h)b(gh)^{-1}$.

The groupoid multiplication on $\widetilde{\Gamma}$ and the 2-cocycle c are related by the following equation

$$(g, \lambda_1)(h, \lambda_2) = \big(gh, \lambda_1\lambda_2 c(g,h)\big), \tag{2}$$

where we identify $\widetilde{\Gamma}$ with $\Gamma \times S^1$ by choosing a trivialization.

Remark 2.8. Let us explain in the language of groupoids what Ruan calls discrete torsion in [30]. By definition [30, Definition 4.6], a discrete torsion is a cohomology class in $H^2(\pi_1^{\mathrm{orb}}(\mathfrak{X}), S^1)$. One can show that the group $\pi_1^{\mathrm{orb}}(\mathfrak{X})$ satisfies the following universal property: for any discrete group G, any generalized morphism $\Gamma \to G$ factorizes through the canonical generalized morphism $\Gamma \to \pi_1^{\mathrm{orb}}(\mathfrak{X})$. Therefore, any discrete torsion determines a class in $H^2(\mathfrak{X}, \mathcal{S}^1) \cong H^3(\mathfrak{X}, \mathbb{Z})$, and a class in $H^3(\mathfrak{X}, \mathbb{Z})$ comes from a discrete torsion if and only if it comes from the pull-back of an S^1-central extension of a discrete group G by a generalized morphism $\Gamma \to G$.

2.3. *Inertia groupoid*

Let $\Gamma \rightrightarrows M$ be a proper and étale groupoid representing an orbifold $\mathfrak{X}$. Let $S\Gamma = \{g \in \Gamma \mid s(g) = t(g)\}$ be the space of closed loops. Then $S\Gamma$ is a manifold, on which the natural action of Γ by conjugation is smooth. Thus one may form the transformation groupoid $\Lambda\Gamma : S\Gamma \rtimes \Gamma \rightrightarrows S\Gamma$, which is called the *inertia groupoid*. Its Morita equivalence class $\Lambda\mathfrak{X}$ is called the *inertia orbifold* [24].

If $S^1 \to \widetilde{\Gamma} \to \Gamma \rightrightarrows M$ is an S^1-central extension, then the restriction $\widetilde{\Gamma}' \to S\Gamma$ of this S^1-bundle to $S\Gamma$ is naturally endowed with an action of Γ. To see this, for any $g \in \Gamma$, let $\tilde{g} \in \widetilde{\Gamma}$ be any of its lifting. Then for any $\gamma \in \widetilde{\Gamma}'$ such that $s(\gamma) = t(\tilde{g})$, set

$$\gamma \cdot g = \tilde{g}^{-1}\gamma\tilde{g}. \tag{3}$$

It is simple to see that this Γ-action is well defined, i.e. independent of the choice of the lifting $\tilde{g}$. Thus $\widetilde{\Gamma}' \to S\Gamma$ naturally becomes an S^1-bundle over the inertia orbifold $\Lambda\Gamma \rightrightarrows S\Gamma$ (see also [22, Lemma 6.4.1]). Note that indeed we have $\widetilde{\Gamma}' \cong S\widetilde{\Gamma}$.

Proposition 2.9. *Let $\Gamma \rightrightarrows M$ be an étale and proper groupoid. Then any S^1-central extension $S^1 \to \widetilde{\Gamma} \to \Gamma \rightrightarrows M$ determines an S^1-bundle, and thus a line bundle L', over the inertia groupoid $\Lambda\Gamma \rightrightarrows S\Gamma$.*

Example 2.10. Suppose $\mathfrak{X} = M/G$, where G is a finite group acting on a manifold M by diffeomorphisms. The inertia groupoid is $\hat{M} \rtimes G$, where $\hat{M} = \bigcup_{g\in G} M^g \times \{g\}$ and G acts on $\hat{M}$ by

$$(x, g) \cdot \gamma = \big(x\gamma, \gamma^{-1}g\gamma\big).$$

If $S^1 \to \widetilde{G} \to G$ is an S^1-central extension, then the induced line bundle over the inertia orbifold is a local inner system in the sense of Ruan (see [30], [22, Proposition 4.3.2]).

2.4. Twisted K-theory of orbifolds

Let $\mathfrak{X}$ be an orbifold and $\alpha \in H^3(\mathfrak{X}, \mathbb{Z})$. Choose an S^1-central extension $S^1 \to \widetilde{\Gamma} \to \Gamma \rightrightarrows M$ representing α, and denote as above $L = \widetilde{\Gamma} \times_{S^1} \mathbb{C}$. The space $C_c^\infty(\Gamma, L)$ of smooth, compactly supported sections of the line bundle $L \to \Gamma$ is endowed with the convolution product

$$(f_1 * f_2)(g) = \sum_{h \in \Gamma^{t(g)}} f_1(h) \cdot f_2\big(h^{-1}g\big),$$

and the adjoint

$$\xi^*(g) = \big(\xi\big(g^{-1}\big)\big)^*,$$

where $f_1(h) \cdot f_2(h^{-1}g)$ is computed using the product $L_h \otimes L_{h^{-1}g} \to L_g$.

For all $x \in M$, let $\mathcal{H}_x$ be the Hilbert space obtained by completing $C_c^\infty(\Gamma, L)$ with respect to the scalar product

$$\langle \xi, \eta \rangle = (\xi^* * \eta)(x) = \sum_{g \in \Gamma_x} \big\langle \xi(g), \eta(g) \big\rangle.$$

Let $(\pi_x(f))\xi = f * \xi$ ($f \in C_c^\infty(\Gamma, L)$, $\xi \in \mathcal{H}_x$). Then $f \mapsto \pi_x(f)$ is a $*$-representation of $C_c^\infty(\Gamma, L)$ in $\mathcal{H}_x$. The C^*-algebra $C_r^*(\Gamma, L)$ is, by definition, the completion of $C_c^\infty(\Gamma, L)$ with respect to the norm: $\sup_{x \in M} \|\pi_x(f)\|$ [32]. Its Morita equivalence class does not depend on the choice of the presentation $S^1 \to \widetilde{\Gamma} \to \Gamma \rightrightarrows M$, and therefore its K-theory is uniquely determined:

Definition 2.11. [32] The twisted K-theory group $K_\alpha^i(\mathfrak{X})$ is defined as $K_i(C_r^*(\Gamma, L))$.

Remark 2.12. Another way to see the C^*-algebra $C_r^*(\Gamma, L)$ is as follows. The cohomology class α determines a unique Morita equivalence class of Γ-equivariant bundles of C^*-algebras over M satisfying Fell's condition [19] such that each fiber is isomorphic to the algebra of compact operators. Denote by A_α one of these C^*-algebras, then $C_r^*(\Gamma, L)$ is Morita equivalent to the crossed-product algebra $A_\alpha \rtimes_r \Gamma$.

Remark 2.13. There is another definition of twisted K-theory as the Grothendieck group of twisted vector bundles $K^0_{\alpha,\mathrm{vb}}(\mathfrak{X})$ [1]. The group $K^0_{\alpha,\mathrm{vb}}(\mathfrak{X})$ is always zero when α is not a torsion class. In [32], it is conjectured that the canonical map

$$K^0_{\alpha,\mathrm{vb}}(\mathfrak{X}) \to K^0_\alpha(\mathfrak{X})$$

is an isomorphism when α is torsion. From [32], this conjecture is known to be true in some special cases, such as

(a) $\mathfrak{X}$ is a compact global quotient orbifold M/G (where G is a compact Lie group), and there exists a twisted vector bundle; or
(b) $\mathfrak{X}$ is a compact manifold.

3. Twisted cohomology

This section is devoted to the introduction of twisted cohomology of an orbifold. In the case of discrete torsion, this is introduced by Ruan [30]. Our definition here is, in a certain sense, a combination of discrete torsion case with the twisted cohomology of manifolds.

3.1. De Rham cohomology

First, let us recall the definition of the de Rham double complex of a Lie groupoid. Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Define for all $p \geqslant 0$

$$\Gamma_p = \underbrace{\Gamma \times_M \cdots \times_M \Gamma}_{p \text{ times}},$$

i.e. Γ_p is the manifold of composable sequences of p arrows in the groupoid $\Gamma \rightrightarrows M$ (and $\Gamma_0 = M$). We have $p+1$ canonical maps $\Gamma_p \to \Gamma_{p-1}$ giving rise to a diagram

$$\cdots \Gamma_2 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \Gamma_1 \rightrightarrows \Gamma_0 . \tag{4}$$

Then $\Gamma_\bullet$ is a simplicial manifold.

Consider the double complex $\Omega^*(\Gamma_\bullet)$:

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\Omega^1(\Gamma_0) & \xrightarrow{\partial} & \Omega^1(\Gamma_1) & \xrightarrow{\partial} & \Omega^1(\Gamma_2) & \xrightarrow{\partial} & \cdots \\
d\uparrow & & d\uparrow & & d\uparrow & & \\
\Omega^0(\Gamma_0) & \xrightarrow{\partial} & \Omega^0(\Gamma_1) & \xrightarrow{\partial} & \Omega^0(\Gamma_2) & \xrightarrow{\partial} & \cdots .
\end{array} \tag{5}$$

Its boundary maps are $d : \Omega^k(\Gamma_p) \to \Omega^{k+1}(\Gamma_p)$, the usual exterior differential of differential forms and $\partial : \Omega^k(\Gamma_p) \to \Omega^k(\Gamma_{p+1})$, the alternating sum of the pull-back maps of (4). We denote the total differential by $\delta = (-1)^p d + \partial$. The cohomology groups of the total complex $C^*_{\mathrm{dR}}(\Gamma_\bullet)$

$$H^k_{\mathrm{dR}}(\Gamma_\bullet) = H^k\big(\Omega^\bullet(\Gamma_\bullet)\big)$$

are called the *de Rham cohomology* groups of $\Gamma \rightrightarrows M$.

Assume that $\Gamma \rightrightarrows M$ is an étale proper Lie groupoid representing an orbifold $\mathfrak{X}$. Since the source and target maps are local diffeomorphisms, any point $g \in \Gamma$ induces a local diffeomorphism from an open neighborhood of $s(g)$ to an open neighborhood of $t(g)$. By $\Omega^k(M)^\Gamma$, we denote the space of all differential k-forms on M invariant under such induced actions of all $g \in \Gamma$. Note that the space $\Omega^k(M)^\Gamma$ is independent of the presentation $\Gamma \rightrightarrows M$, so we also denote it by $\Omega^k(\mathfrak{X})$ interchangeably in the sequel. Also it is simple to see that $\bigoplus_k \Omega^k(M)^\Gamma$ is stable under the de Rham differential, thus one obtains a cochain complex $(\Omega^*(M)^\Gamma, d)$, whose cohomology groups are denoted by $H^*_{\mathrm{dR}}(M)^\Gamma$.

Lemma 3.1. *Let $\Gamma \rightrightarrows M$ be an étale proper Lie groupoid. Consider the double complex* (5). *Then we have*:

(1) $E_1^{k,0} = H^k_{\mathrm{dR}}(M)^\Gamma$;
(2) $E_1^{k,p} = 0$ *if* $p \geqslant 1$.

Proof. If $\Gamma \rightrightarrows M$ is étale, the qth row $(\Omega^{q-1}(\Gamma_p), \partial)_{p\geqslant 0}$ is the complex which computes the differentiable groupoid cohomology of Γ with values in the Γ-vector bundle $\bigwedge^{q-1} T^*M \to M$. Thus it is acyclic since $\Gamma \rightrightarrows M$ is proper [12]. □

As an immediate consequence, we obtain the following

Corollary 3.2. *Let $\Gamma \rightrightarrows M$ be an étale proper Lie groupoid representing an orbifold $\mathfrak{X}$. Then*

$$H^*_{\mathrm{dR}}(\mathfrak{X}) \cong H^*_{\mathrm{dR}}(M)^\Gamma.$$

3.2. Connections and the Dixmier–Douady class

Let $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows M$ be an S^1-central extension representing an S^1-gerbe $\widetilde{\mathfrak{X}}$ over an orbifold $\mathfrak{X}$. By a *pseudo-connection* [4,5], we mean a pair (θ, B), where $\theta \in \Omega^1(\widetilde{\Gamma})$ is a connection 1-form for the bundle $\widetilde{\Gamma} \to \Gamma$, and $B \in \Omega^2(M)$ is a 2-form. It is simple to check that $\delta(\theta + B) \in Z^3_{\mathrm{dR}}(\widetilde{\Gamma}_\bullet)$ descends to $Z^3_{\mathrm{dR}}(\Gamma_\bullet)$, i.e. there exist unique $\eta \in \Omega^1(\Gamma_2)$, $\omega \in \Omega^2(\Gamma)$ and $\Omega \in \Omega^3(M)$ such that

$$\delta(\theta + B) = \pi^*(\eta + \omega + \Omega).$$

Then $\eta + \omega + \Omega$ is called the *pseudo-curvature* of the pseudo-connection $\theta + B$. It is known [4,5] that the class $[\eta + \omega + \Omega] \in H^3_{\mathrm{dR}}(\Gamma_\bullet)$ is independent of the choice of pseudo-connections. Under the canonical homomorphism

$$H^3(\mathfrak{X}, \mathbb{Z}) \to H^3(\mathfrak{X}, \mathbb{R}) \cong H^3_{\mathrm{dR}}(\Gamma_\bullet),$$

the Dixmier–Douady class of $\widetilde{\mathfrak{X}}$ maps to $[\eta + \omega + \Omega]$.

Following [7], we introduce

Definition 3.3. [5] Given an S^1-central extension $S^1 \to \widetilde{\Gamma} \to \Gamma \rightrightarrows M$ of an étale proper groupoid $\Gamma \rightrightarrows M$,

(i) a connection 1-form $\theta \in \Omega^1(\widetilde{\Gamma})$ for the S^1-bundle $\widetilde{\Gamma} \to \Gamma$, such that $\partial\theta = 0$, is a *connection* on $\widetilde{\Gamma}_\bullet \to \Gamma_\bullet$;
(ii) given θ, a 2-form $B \in \Omega^2(M)$, such that $d\theta = \partial B$, is a *curving* on $\widetilde{\Gamma}_\bullet \to \Gamma_\bullet$;
(iii) and given (θ, B), the 3-form $\Omega = dB \in \Omega^3(M)^\Gamma$ is called the 3-*curvature* of $(\widetilde{\Gamma}, \theta, B)$;
(iv) $\widetilde{\Gamma}_\bullet \to \Gamma_\bullet$ is called a *flat* gerbe, if furthermore $\Omega = 0$.

It is clear that Ω is a closed Γ-invariant 3-form, and thus defines a class in $H^3_{\mathrm{dR}}(M)^{\Gamma}$. In fact, by the discussion above, it is simple to see that $[\Omega] \in H^3_{\mathrm{dR}}(M)^{\Gamma}$ is the image of the Dixmier–Douady class of the S^1-gerbe $\widetilde{\mathfrak{X}}$ over $\mathfrak{X}$ under the canonical homomorphism

$$H^3(\mathfrak{X}, \mathbb{Z}) \to H^3(\mathfrak{X}, \mathbb{R}) \cong H^3_{\mathrm{dR}}(\Gamma_\bullet) \cong H^3_{\mathrm{dR}}(M)^{\Gamma}.$$

Remark 3.4. Assume that $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$ is an S^1-central extension whose corresponding Dixmier–Douady class is α. Then the following are equivalent:

(i) α is a torsion class;
(ii) there exists a flat connection.

Lemma 3.5. *Given an S^1-central extension of Lie groupoids $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows M$, assume that $\theta \in \Omega^1(\widetilde{\Gamma})$ is a connection one-form for the S^1-principal bundle $\pi : \widetilde{\Gamma} \to \Gamma$. Suppose we are given a trivialization $\widetilde{\Gamma} \cong \Gamma \times S^1$, and denote by $\nabla = d + 2\pi i\phi$ ($\phi \in \Omega^1(\Gamma)$) the associated covariant differential on the line bundle $\widetilde{\Gamma} \times_{S^1} \mathbb{C} \cong \Gamma \times \mathbb{C}$. Then*:

(a) *$\theta = dt + \pi^*\phi$, where dt is the pull-back of the one-form $dt \in \Omega^1(\mathbb{R})$ to $S^1 \cong \mathbb{R}/\mathbb{Z}$.*
(b) *Let $c : \Gamma_2 \to S^1$ be the groupoid 2-cocycle associated to the central extension $S^1 \to \widetilde{\Gamma} \to \Gamma$. Then $\partial\theta = \pi^*(\partial\phi - \frac{dc}{2\pi i c})$.*

Therefore θ is a connection for $\widetilde{\Gamma}$ if and only if $\partial\phi = \frac{dc}{2\pi i c}$.

Proof. (a) is trivial. To show (b), recall that the multiplication on the groupoid $\widetilde{\Gamma} \rightrightarrows M$ is given by $(g, s)(h, t) = (gh, s + t + \tilde{c}(g, h)) \in \Gamma \times \mathbb{R}/\mathbb{Z}$, where $c = e^{2\pi i \tilde{c}}$. It thus follows that

$$\partial\theta|_{((g,s)(h,t))} = \partial\phi|_{(g,h)} + ds + dt - d\big(s + t + \tilde{c}(g, h)\big) = \partial\phi|_{(g,h)} - \frac{dc}{2\pi i c}(g, h). \qquad \square$$

The following proposition indicates that, by passing to a nice refinement, connections with curving always exist for any S^1-central extension over an étale proper groupoid.

Proposition 3.6. *Let $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ be an S^1-gerbe over an orbifold $\mathfrak{X}$, and $\Gamma \rightrightarrows M$ a nice étale proper groupoid representing $\mathfrak{X}$. Then $\widetilde{\mathfrak{X}}$ is represented by an S^1-central extension $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$, which admits a connection and curving.*

Proof. Choose a trivialization of the principal bundle $\widetilde{\Gamma} \to \Gamma$. Let $c : \Gamma_2 \to S^1$ be the groupoid 2-cocycle associated to the S^1-central extension. Consider the equation

$$\partial\phi = \frac{dc}{2\pi i c} = \frac{1}{2\pi i}\, d\ln c,$$

where $\phi \in \Omega^1(\Gamma)$. Since $\partial d \ln c = d\partial \ln c = 0$, the equation above always admits a solution according to [12]. Thus $\theta = dt + \pi^*\phi$ is a connection for $\widetilde{\Gamma}_\bullet \to \Gamma_\bullet$ according to Lemma 3.5. Now since $\partial d\phi = d\partial\phi = \frac{1}{2\pi i}\, d^2 \ln c = 0$, again according to [12] there exists a $B \in \Omega^2(M)$ such that $d\phi = \partial B$. Therefore $d\theta = \pi^*\, d\phi = \partial B$. This shows that (θ, B) is a curving. The 3-curvature is thus given by $\Omega = dB \in \Omega^3(M)^{\Gamma}$. $\square$

Remark 3.7. It is simple to see that the forms $-\frac{dc}{c} \in \Omega^1(\Gamma_2)$, $-d\phi \in \Omega^2(\Gamma)$ and $\Omega \in \Omega^3(M)^\Gamma$ all represent the same cohomology class $\alpha \in H^3(\mathfrak{X}, \mathbb{R})$.

Proposition 3.8. *Under the same hypothesis of Proposition* 3.6*, the space of connections is an affine space with underlying vector space $\Omega^1(M)/\Omega^1(M)^\Gamma$.*

Proof. From Lemma 3.5, it follows that $\theta' = dt + \pi^*\phi'$ is another connection if and only if $\partial(\phi' - \phi) = 0$, i.e. $\phi' - \phi \in \ker\partial$. Since the complex

$$\Omega^1(M) \xrightarrow{\partial} \Omega^1(\Gamma_1) \xrightarrow{\partial} \Omega^1(\Gamma_2) \xrightarrow{\partial} \cdots$$

is acyclic, we see that $\ker\partial \cong \Omega^1(M)/\Omega^1(M)^\Gamma$. □

Assume that $P_0 \to X_0$ is an S^1-bundle over a Lie groupoid $X_1 \rightrightarrows X_0$. Recall that [5,20] a *connection* is a connection 1-form θ on P_0 (as an S^1-bundle over X_0) such that $\partial\theta = 0$. A connection is said to be *flat* if moreover $d\theta = 0$. Here $\partial\theta = s^*\theta - t^*\theta \in \Omega^1(P_1)$, and $P_1 \rightrightarrows P_0$ is the transformation groupoid $X_1 \times_{X_0} P_0 \rightrightarrows P_0$. As it was shown in [20], equivalently a connection is a connection 1-form θ on P_0 which is basic with respect to the action of the pseudo-group of local bisections of $X_1 \rightrightarrows X_0$. In particular, if $X_1 \rightrightarrows X_0$ is an étale proper groupoid, the latter is equivalent to that θ is invariant under the X_1-action.

We end this subsection with the following

Proposition 3.9. *Let $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows M$ be an S^1-central extension of groupoids, which is assumed to admit a connection. Then there exists an induced connection on its associated S^1-bundle $\widetilde{\Gamma}' \to S\Gamma$ over the inertia groupoid $\Lambda\Gamma$ constructed as in Proposition* 2.9*. It is flat if furthermore the connection admits a curving.*

Moreover, if $\Gamma \rightrightarrows M$ is an étale proper groupoid, this induced connection is unique, i.e., is independent of the choices of connections on $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows M$.

Proof. Assume that $\theta \in \Omega^1(\widetilde{\Gamma})$ is a connection of the S^1-central extension $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$. We define $\theta' \in \Omega^1(\widetilde{\Gamma}')$ as the pull back of θ to $\widetilde{\Gamma}'$. Note that $\theta \in \Omega^1(\widetilde{\Gamma})$ is a connection if and only if it satisfies the property that for any horizontal path $t \mapsto (g_t, h_t) \in \widetilde{\Gamma}_2$, the path $t \mapsto g_t h_t$ is again horizontal. In order to show that $\theta' \in \Omega^1(\widetilde{\Gamma}')$ defines a connection for the S^1-bundle $\widetilde{\Gamma}' \to S\Gamma$ over the inertia groupoid $\Lambda\Gamma$, it suffices to show that for any horizontal path $t \mapsto h_t \in \widetilde{\Gamma}'$ and any path $t \mapsto \gamma_t \in \Gamma$ such that h_t and γ_t are composable, the path $h_t \cdot \gamma_t$ is again horizontal. Recall that the action of the element $\gamma_t \in \Gamma$ on $h_t \in \widetilde{\Gamma}'$ is defined by $h_t \cdot \gamma_t = \tilde{\gamma}_t^{-1} h_t \tilde{\gamma}_t$, where $\tilde{\gamma}_t$ is any lifting of γ_t. Let us choose $\tilde{\gamma}_t$ such that $t \mapsto \tilde{\gamma}_t$ is a horizontal lifting of $t \mapsto \gamma_t$. It is then clear that $t \mapsto \tilde{\gamma}_t^{-1} h_t \tilde{\gamma}_t$ is a horizontal path. This proves the first assertion.

Assume that the connection $\theta \in \Omega^1(\widetilde{\Gamma})$ admits a curving. That is, $d\theta = \partial B$ for some $B \in \Omega^2(M)$. Therefore, we have $d\theta' = \partial B_{|S\Gamma} = 0$. That is, $\widetilde{\Gamma}' \to S\Gamma$ is a flat bundle over $\Lambda\Gamma$.

Now assume that $\Gamma \rightrightarrows M$ is étale and proper. Note that any two connections always differ by $\pi^*\alpha$, where $\alpha \in \Omega^1(\Gamma)$ is a one-form satisfying $\partial\alpha = 0$. According to Lemma 3.1(2), we have $\alpha = \partial A$, where $A \in \Omega^1(M)$. Since $(\partial A)_{|S\Gamma} = 0$, θ' is independent of the choice of connections, and is therefore canonically defined. □

3.3. Twisted cohomology

We are now ready to introduce the twisted cohomology of an orbifold. If E is a vector bundle over an orbifold $\mathfrak{X}$, by $\Omega^*(\mathfrak{X}, E)$ we denote the space of Γ-invariant E-valued differential forms $\Omega^*(M, E)^\Gamma$, where $\Gamma \rightrightarrows M$ is a presentation of $\mathfrak{X}$. Note that $\Omega^*(M, E)^\Gamma$ is independent of the choice of presentations, so this justifies the notation. Similarly, $\Omega_c^*(\mathfrak{X}, E)$ denotes the subspace of $\Omega^*(M, E)^\Gamma$ consisting of differential forms whose supports are compact in the orbit space $|\Gamma|$.

Let $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ be an S^1-gerbe over an orbifold $\mathfrak{X}$ with Dixmier–Douady class $\alpha \in H^3(\mathfrak{X}, \mathbb{Z})$. Let $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$ be an S^1-central extension representing this gerbe, which admits a connection θ, curving B and 3-curvature $\Omega \in \Omega^3(M)^\Gamma$. Denote by $L' \to S\Gamma$ the induced complex line bundle over the inertia groupoid $\Lambda\Gamma \rightrightarrows S\Gamma$, which admits a canonical flat connection according to Proposition 3.9. By $\nabla' : \Omega_c^*(S\Gamma, L') \to \Omega_c^{*+1}(S\Gamma, L')$ we denote its corresponding covariant differential. It is clear that $\Omega_c^*(S\Gamma, L')^\Gamma$ is stable under ∇' since ∇' is Γ-invariant.

Definition 3.10. We define the twisted cohomology groups (with compact supports) $H_c^*(\mathfrak{X}, \alpha)$ as the cohomology of the complex

$$\big(\Omega_c^*(\Lambda\mathfrak{X}, L')((u)), \nabla' - 2\pi i \Omega u \wedge \cdot\big),$$

where u is a formal variable of degree -2, and $((u))$ stands for formal Laurent series in u.

Note that $\nabla' - 2\pi i \Omega u \wedge \cdot$ is indeed a differential since $(\nabla')^2 = 0$, $\Omega \wedge \Omega = 0$ and $d\Omega = 0$.

Remark 3.11.

(1) In the definition above, $2\pi i \Omega$ can be replaced by any non-zero multiple of Ω, for instance Ω. Indeed, the complex above is isomorphic to $(\Omega_c^*(\Lambda\mathfrak{X}, L')((u')), \nabla' - u'\Omega \wedge \cdot)$ by letting $u = \frac{u'}{2\pi i}$.
(2) If $\Omega' \in \Omega^3(M)^\Gamma$ is another 3-curvature as above, then $(\Omega_c^*(\Lambda\mathfrak{X}, L')((u)), \nabla' - 2\pi i \Omega u \wedge \cdot)$ and $(\Omega_c^*(\Lambda\mathfrak{X}, L')((u)), \nabla' - 2\pi i \Omega' u \wedge \cdot)$ are isomorphic as chain complexes. However the isomorphism is not canonical. The induced map in cohomology is unique up to an automorphism of the form $\omega \mapsto e^{u\beta}\omega$, where $\beta \in H^2(\mathfrak{X}, \mathbb{R})$. To specify which isomorphism to use, one needs to specify the connections. This statement will be made more precise below (Proposition 4.8).
(3) Also, recall that $\Omega_c^*(\Lambda\mathfrak{X}, L')$ is, by definition, $\Omega_c^*(S\Gamma, L')^\Gamma$. The latter does not depend on the choice of the presentation $\Gamma \rightrightarrows M$.

In conclusion, the twisted cohomology groups defined above only depend on the orbifold $\mathfrak{X}$ and the class $\alpha \in H^3(\mathfrak{X}, \mathbb{Z})$.

4. Non-commutative Chern character

4.1. The convolution algebra $\Omega_c^*(\Gamma, L)$

Assume that $\Gamma \rightrightarrows M$ is an étale proper groupoid and $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$ is an S^1-central extension. Let $L \to \Gamma$ be its associated complex line bundle. By $\Omega_c^*(\Gamma, L)$, we denote the space of all compactly supported smooth sections of the vector bundle $\bigwedge T^*\Gamma \otimes L \to \Gamma$.

Lemma 4.1. *$\Omega_c^*(\Gamma, L)$ admits a convolution product, which makes it into a graded associative algebra.*

Proof. To see this, note that given a vector bundle $E \to \Gamma$ equipped with an associative product $E_g \otimes E_h \to E_{gh}$, one can define a convolution product on the space of compactly supported smooth sections $C_c^\infty(\Gamma, E)$ by the formula

$$(f_1 * f_2)(g) = \sum_{h \in \Gamma^{t(g)}} f_1(h) \cdot f_2\big(h^{-1}g\big).$$

Now consider $E = \bigwedge T^*\Gamma \otimes L$. Given any $(g, h) \in \Gamma_2$ and $\omega_1 \otimes \xi_1 \in \bigwedge T_g^*\Gamma \otimes L_g$, $\omega_2 \otimes \xi_2 \in \bigwedge T_h^*\Gamma \otimes L_h$, define

$$(\omega_1 \otimes \xi_1) \cdot (\omega_2 \otimes \xi_2) = (r_h\omega_1 \wedge l_g\omega_2) \otimes \xi_1 \cdot \xi_2,$$

where l_g and r_h denote the left and right actions of Γ on its exterior product of cotangent bundles, which are well defined since Γ is étale. It is simple to see that this indeed defines an associative product $E_g \otimes E_h \to E_{gh}$.

By $\Omega_c^*(\Gamma, L)$, we denote the space of the compactly supported smooth sections of this bundle. Thus it admits a convolution product, which is denoted by $*$. □

Note that since M is an open and closed submanifold of Γ and the restriction of L to M is the trivial line bundle, $\Omega^*(M)$ can be naturally considered as a subspace of $\Omega^*(\Gamma, L)$. Moreover, the convolution product between elements in $\Omega^*(M)$ and elements in $\Omega^*(\Gamma, L)$ are well defined using exactly the same formula above, even if they may not be compactly supported. The following lemma describes the convolution product between elements in these two spaces.

Lemma 4.2. *For any $B \in \Omega^*(M)$ and $\omega \in \Omega^*(\Gamma, L)$, we have*:

(i) $B * \omega = t^*B \wedge \omega$;
(ii) $\omega * B = (-1)^{|B|.|\omega|} s^*B \wedge \omega$;
(iii) $[B, \omega] = -\partial B \wedge \omega$.

Proof. This follows from a straightforward verification. □

Now let $\theta \in \Omega^1(\widetilde{\Gamma})$ be a connection 1-form on the principal S^1-bundle $\widetilde{\Gamma} \to \Gamma$, which induces a linear connection on the associated line bundle $L \to \Gamma$. By $\nabla : \Omega^*(\Gamma, L) \to \Omega^{*+1}(\Gamma, L)$, we denote its corresponding covariant differential.

Lemma 4.3. *Under the same hypothesis as above, θ is a connection for the S^1-central extension $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$ if and only if the following identities are satisfied*:

(i) $\nabla(f_1 * f_2) = f_1 * \nabla f_2 + (\nabla f_1) * f_2$, $\forall f_1, f_2 \in C_c^\infty(\Gamma, L)$;
(ii) $\nabla f = df$ *for all* $f \in C_c^\infty(M, \mathbb{C})$, *where* $C_c^\infty(M, \mathbb{C}) \cong C_c^\infty(M, L)$ *is considered as a subspace of* $C_c^\infty(\Gamma, L)$ *and, similarly,* $\Omega_c^1(M, \mathbb{C})$ *is a subspace of* $\Omega_c^1(\Gamma, L)$.

Proof. We show the direct implication (the converse is proved by working backwards).

Let us show that (i) holds at any point $k_0 \in \Gamma$. Since $\Gamma \rightrightarrows M$ is proper, $F = \Gamma^{t(k_0)} \cap \operatorname{supp}(f_1)$ is finite. Let $\varphi_g \in C_c^\infty(\Gamma)$ $(g \in F)$ such that the φ_g's have disjoint support and $\varphi_g = 1$ in a neighborhood of g. Then the evaluation of both sides of (i) at the point k_0 remains unchanged if we replace f_1 by $\sum_g \varphi_g f_1$. By linearity, we may assume that f_1 is of the form $\varphi_{g_0} f_1$ for some g_0, i.e. f_1 is supported on a small neighborhood of g_0. Similarly, f_2 can be assumed to be supported on a small neighborhood of h_0 where $g_0 h_0 = k_0$.

Let $g \mapsto \tilde{g}$ (respectively $h \mapsto \tilde{h}$) be a local section of $\widetilde{\Gamma} \to \Gamma$ around g_0 (respectively h_0). By Leibniz' rule, we can assume that $f_1(g) = \tilde{g}$ for g close to g_0 and $f_2(h) = \tilde{h}$ for h close to h_0. Let $t \mapsto (g(t), h(t)) \in \Gamma_2$ be a smooth path with $(g(0), h(0)) = (g_0, h_0)$. Then

$$(f_1 * \nabla f_2)(g_0 h_0)\big((gh)'(0)\big) = 2\pi i \theta(\tilde{h}_0)\big(f_2'(h_0)\big(h'(0)\big)\big)\tilde{g}_0\tilde{h}_0,$$
$$(\nabla f_1 * f_2)(g_0 h_0)\big((gh)'(0)\big) = 2\pi i \theta(\tilde{g}_0)\big(f_1'(g_0)\big(g'(0)\big)\big)\tilde{g}_0\tilde{h}_0,$$
$$\big(\nabla(f_1 * f_2)\big)(g_0 h_0)\big((gh)'(0)\big) = 2\pi i \theta(\tilde{g}_0\tilde{h}_0)\big(m_*\big(f_1'(g_0)\big(g'(0)\big), f_2'(h_0)\big(h'(0)\big)\big)\big)\tilde{g}_0\tilde{h}_0,$$

where $m : \widetilde{\Gamma}_2 \to \widetilde{\Gamma}$ is the multiplication. Let us explain, for instance, the notations in the first formula above. $(f_1 * \nabla f_2)(g_0 h_0)$ is an element of $T^*_{g_0 h_0}\Gamma \otimes L_{g_0 h_0}$, so its evaluation on the vector $(gh)'(0)$ is an element of $L_{g_0 h_0} = L_{k_0}$. On the right-hand side, the evaluation of $\theta(\tilde{h}_0) \in T^*_{\tilde{h}_0}\widetilde{\Gamma}$ on the vector $f_2'(h_0)(h'(0))$ is a scalar; multiplying this scalar by $\tilde{g}_0\tilde{h}_0 \in \pi^{-1}(k_0) \subset \widetilde{\Gamma} \subset L$ yields an element of L_{k_0}.

Therefore, we get

$$\begin{aligned}&\big(\nabla(f_1 * f_2) - \nabla f_1 * f_2 - f_1 * \nabla f_2\big)(g_0 h_0)\big((gh)'(0)\big)\\ &\quad = -2\pi i(\partial\theta)(\tilde{g}_0, \tilde{h}_0)\big(f_1'(g_0)\big(g'(0)\big), f_2'(h_0)\big(h'(0)\big)\big) = 0.\end{aligned}$$

To show (ii), note that (i) obviously holds whenever $f_j \in C^\infty(M)$ (not necessarily compactly supported). For $f_1 = f_2 = 1_M$, we have $\nabla(1_M) = 0$. Then, by Leibniz' identity, we get $\nabla(f) = \nabla(f.1_M) = df\,1_M + f\nabla(1_M) = df$. $\square$

As a consequence, we have

Proposition 4.4. $\nabla : \Omega_c^*(\Gamma, L) \to \Omega_c^{*+1}(\Gamma, L)$ *satisfies the Leibniz identity*

$$\nabla(\omega_1 * \omega_2) = (\nabla\omega_1) * \omega_2 + (-1)^{|\omega_1|}\omega_1 * (\nabla\omega_2). \tag{6}$$

Proof. It is simple to see, by using Lemma 4.2, that Eq. (6) holds when $\omega_1 \in C_c^\infty(M, \mathbb{C})$ and $\omega_2 \in \Omega_c^*(\Gamma, L)$.

To prove Eq. (6) in general, it suffices to prove it locally. By linearity, we may assume that for $j = 1, 2$, $\omega_j = \eta_j * f_j$ $(\eta_j \in \Omega_c^*(M, \mathbb{C}),\ f_j \in C_c^\infty(\Gamma, L))$, with f_j supported on a small neighborhood of g_j and η_j supported on a small neighborhood of $t(g_j)$, where $(g_1, g_2) \in \Gamma_2$. Since Γ is étale, there is a canonical local bisection $\mathcal{L}$ of Γ through the point g_1. It is clear that $\mathcal{L}$ induces a local diffeomorphism on M, denoted by $Ad_{\mathcal{L}^{-1}}$, from a small neighborhood of $t(g_1)$ to a small neighborhood of $s(g_1)$. From Lemma 4.2, it follows that

$$\omega * \eta = (-1)^{|\omega||\eta|}\tilde{\eta} * \omega, \tag{7}$$

for any $\eta \in \Omega_c^*(M, \mathbb{C})$ supported around $s(g_1)$, and $\omega \in \Omega_c^*(\Gamma, L)$ supported around g_1, where $\tilde{\eta} = Ad^*_{\mathcal{L}^{-1}}\eta$. Therefore, we have

$$\begin{aligned}
\nabla(\omega_1 * \omega_2) &= \nabla(\eta_1 * \tilde{\eta}_2 * f_1 * f_2) \\
&= d(\eta_1 * \tilde{\eta}_2) * f_1 * f_2 + (-1)^{|\omega_1|+|\omega_2|}\eta_1 * \tilde{\eta}_2 * \nabla(f_1 * f_2) \\
&= (d\eta_1) * \tilde{\eta}_2 * f_1 * f_2 + (-1)^{|\omega_1|}\eta_1 * d\tilde{\eta}_2 * f_1 * f_2 \\
&\quad + (-1)^{|\omega_1|+|\omega_2|}\eta_1 * \tilde{\eta}_2 * (\nabla f_1) * f_2 + (-1)^{|\omega_1|+|\omega_2|}\eta_1 * \tilde{\eta}_2 * f_1 * \nabla f_2 \\
&= (d\eta_1) * f_1 * \eta_2 * f_2 + (-1)^{|\omega_1|}\eta_1 * \widetilde{d\eta_2} * f_1 * f_2 \\
&\quad + (-1)^{|\omega_2|}\eta_1 * (\nabla f_1) * \eta_2 * f_2 + (-1)^{|\omega_1|+|\omega_2|}\eta_1 * f_1 * \eta_2 * \nabla f_2 \\
&= (\nabla\omega_1) * \omega_2 + (-1)^{|\omega_1|}\omega_1 * \nabla\omega_2.
\end{aligned}$$

Equation (6) thus follows. □

4.2. *The trace map* $\mathrm{Tr}: \Omega_c^*(\Gamma, L) \to \Omega_c^*(\Lambda\mathfrak{X}, L')$

This subsection is devoted to the introduction of the trace map. Let $\omega \in \Omega_c^*(\Gamma, L)$. For any $h \in S\Gamma$ and $g \in \Gamma$ such that $s(h) = t(g)$, $\omega_{g^{-1}hg} \in \bigwedge T^*_{g^{-1}hg}\Gamma \otimes L_{g^{-1}hg}$. Then

$$i^*\omega_{g^{-1}hg} \in \bigwedge T^*_{g^{-1}hg} S\Gamma \otimes L_{g^{-1}hg} \cong \bigwedge T^*_{g^{-1}hg} S\Gamma \otimes L'_{g^{-1}hg},$$

where $i: S\Gamma \to \Gamma$ is the inclusion. Therefore $g_* i^*\omega_{g^{-1}hg} \in \bigwedge T^*_h S\Gamma \otimes L'_h$. Here $\bigwedge T^*S\Gamma \otimes L' \to S\Gamma$ is considered as a vector bundle over the inertia groupoid $\Lambda\Gamma \rightrightarrows S\Gamma$ and therefore admits a Γ-action, which is denoted by g_*. Now define $\mathrm{Tr}(\omega) \in \Omega_c^*(S\Gamma, L')$ by

$$\mathrm{Tr}(\omega)_h := \sum_{g \in \Gamma^{s(h)}} g_* i^*\omega_{g^{-1}hg}, \quad \forall h \in S\Gamma.$$

It is simple to see that $\mathrm{Tr}(\omega)$ is a Γ-invariant element in $\Omega^*(S\Gamma, L')$ whose support is compact in $|\Lambda\Gamma|$. Therefore it defines an element in $\Omega_c^*(\Lambda\mathfrak{X}, L')$.

The proposition below describes an important property of this trace map.

Proposition 4.5.

(a) $\forall\omega_1 \in \Omega^{|\omega_1|}(\Gamma, L)$ *and* $\omega_2 \in \Omega^{|\omega_2|}(\Gamma, L)$*, we have*

$$\mathrm{Tr}(\omega_1 * \omega_2) = (-1)^{|\omega_1||\omega_2|}\,\mathrm{Tr}(\omega_2 * \omega_1).$$

(b) *Assume that* ∇ *is a linear connection on* $L \to \Gamma$ *induced from a connection on the* S^1*-central extension* $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$. *Let* ∇' *be its induced connection on the line bundle* $L' \to S\Gamma$ *over* $\Lambda\Gamma \rightrightarrows S\Gamma$. *Then*

$$\nabla'(\mathrm{Tr}\,\omega) = \mathrm{Tr}(\nabla\omega).$$

Proof. It is simple to see, from a straightforward computation, that for any $h \in S\Gamma$,

$$\begin{aligned}
\mathrm{Tr}(\omega_1 * \omega_2)_h &= \sum_{g \in \Gamma^{s(h)},\, k_1k_2 = g^{-1}hg} g_* i^*\big(\omega_1(k_1)\cdot\omega_2(k_2)\big), \\
\mathrm{Tr}(\omega_2 * \omega_1)_h &= \sum_{g' \in \Gamma^{s(h)},\, k_2k_1 = g'^{-1}hg'} g'_* i^*\big(\omega_2(k_2)\cdot\omega_1(k_1)\big) \quad (\text{let } g = g'k_2) \\
&= \sum_{g \in \Gamma^{s(h)},\, k_1k_2 = g^{-1}hg} \big(gk_2^{-1}\big)_* i^*\big(\omega_2(k_2)\cdot\omega_1(k_1)\big).
\end{aligned}$$

It thus suffices to prove the following identity:

$$i^*\big(\omega_1(k_1)\cdot\omega_2(k_2)\big) = (-1)^{|\omega_1||\omega_2|} i^*\big(k_2^{-1}\big)_*\big(\omega_2(k_2)\cdot\omega_1(k_1)\big). \tag{8}$$

By linearity, we can assume that $\omega_j = \eta_j \otimes \xi_j \in \bigwedge T^*_{k_j}\Gamma \otimes L_{k_j}$, $j = 1, 2$. Thus Eq. (8) reduces to

$$\xi_1 \cdot \xi_2 = k_{2*}^{-1}(\xi_2 \cdot \xi_1), \quad \text{and} \tag{9}$$

$$i^*(\eta_1 \cdot \eta_2) = (-1)^{|\eta_1||\eta_2|} i^* k_{2*}^{-1}(\eta_2 \cdot \eta_1). \tag{10}$$

Since $\xi_2 \in L_{k_2}$, according to Eq. (3), we have

$$k_{2*}^{-1}(\xi_2 \cdot \xi_1) = \xi_2^{-1} \cdot (\xi_2 \cdot \xi_1) \cdot \xi_2 = \xi_1 \cdot \xi_2.$$

Thus Eq. (9) follows.

For Eq. (10), note that

$$\begin{aligned}
k_{2*}^{-1}(\eta_2 \cdot \eta_1) &= k_{2*}^{-1}\big(r_{k_1}\eta_2(k_2) \wedge l_{k_2}\eta_1(k_1)\big) \\
&= Ad_{k_2^{-1}} r_{k_1}\eta_2(k_2) \wedge Ad_{k_2^{-1}} l_{k_2}\eta_1(k_1) \\
&= (-1)^{|\eta_1||\eta_2|}\big(r_{k_2}\eta_1(k_1) \wedge Ad_{(k_1k_2)^{-1}} l_{k_1}\eta_2(k_2)\big) \\
&= (-1)^{|\eta_1||\eta_2|}\big(r_{k_2}\eta_1(k_1) \wedge (k_1k_2)_*^{-1}\big(l_{k_1}\eta_2(k_2)\big)\big).
\end{aligned}$$

Therefore it suffices to prove that

$$(k_1k_2)_*^{-1} i^* l_{k_1}\eta_2(k_2) = i^* l_{k_1}\eta_2(k_2),$$

since $(k_1k_2)_*^{-1}$ and i^* commute. The latter holds due to the following general fact: for any $\gamma \in S\Gamma$, and $\eta \in \bigwedge T^*_\gamma(S\Gamma)$, one has

$$\gamma_*\eta = \eta. \tag{11}$$

Locally, we can assume that $\Gamma \rightrightarrows M$ is the crossed product $M \rtimes H \rightrightarrows M$ of a manifold by a finite group. We then have $\eta \in \bigwedge T_x^* M^h$ and $r = (x, h)$, where $x \in M^h$ and $h \in H$. Thus Eq. (11) becomes

$$h_* \eta = \eta,$$

which is obvious since h acts on $T_x M^h$ as the identity. □

4.3. Chain map

Now we are ready to prove the main theorem. First we will introduce a chain map from the chain complex of periodic cyclic homology to the chain complex of twisted cohomology.

From now on, we assume that $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$ is an S^1-central extension of a proper étale groupoid $\Gamma \rightrightarrows M$, which admits a connection θ (with the corresponding covariant differential ∇ on the associated line bundle $L \to \Gamma$), curving B and 3-curvature Ω as in Definition 3.3. We start by constructing a chain map from the cyclic bi-complex to the complex $(\Omega^*(\Lambda\mathfrak{X}, L'))((u))$.

Let $\mathcal{A} = C_c^\infty(\Gamma, L)$, and then $\mathcal{A}^{\otimes n} = C_c^\infty(\Gamma^n, L^{\otimes n})$ (here we use the inductive tensor product, for which we have $C_c^\infty(M) \otimes C_c^\infty(N) \cong C_c^\infty(M \times N)$). Denote by $\widetilde{\mathcal{A}}$ the unitization of $\mathcal{A}$.

Let $CC_k(\mathcal{A}) = \widetilde{\mathcal{A}} \otimes \mathcal{A}^{\otimes k}$. Recall [6] [21, Proposition 2.2.16] that the periodic cyclic homology of $\mathcal{A}$ is the cohomology of the complex $(CC_k(\mathcal{A})((u)), b + u\mathrm{B})$, where u is a formal variable of degree -2, and b is a differential of degree -1 while B is a differential of degree 1 defined by

$$\begin{aligned} b(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_k) &= \sum_{j=0}^{k-1} (-1)^j \tilde{a}_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_k \\ &\quad + (-1)^k a_k \tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_{k-1}, \\ \mathrm{B}(\tilde{a}_0 \otimes a_1 \otimes \cdots a_k) &= \sum_{i=0}^{k} (-1)^{ik} 1 \otimes a_i \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{i-1}. \end{aligned}$$

Then b and B satisfy the identity $b^2 = \mathrm{B}^2 = b\mathrm{B} + \mathrm{B}b = 0$.

Following [18] (see also [23]), we introduce a linear map $\tau_{\nabla,B} : CC_k(\mathcal{A}) \to \Omega_c^*(\Lambda\mathfrak{X}, L')((u))$ by

$$\begin{aligned} &\tau_{\nabla,B}(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_k) \\ &\quad = \int_{\Delta^k} \mathrm{Tr}\big(\tilde{a}_0 * e^{2\pi i u s_0 B} * \nabla(a_1) * \cdots * \nabla(a_k) * e^{2\pi i u s_k B}\big)\, ds_1 \cdots ds_k, \end{aligned} \tag{12}$$

where the integration is over the k-simplex $\Delta^k = \{(s_0, \ldots, s_k) \mid s_0 \geqslant 0, \ldots, s_k \geqslant 0, s_0 + \cdots + s_k = 1\}$. Here the curving 2-form $B \in \Omega^2(M)$ is considered as an element of $\Omega^2(\Gamma, L)$ since M is an open and closed submanifold of Γ and the restriction of L to M is the trivial line bundle. We then extend $\tau_{\nabla,B}$ to a $\mathbb{C}((u))$-linear map

$$\tau_{\nabla,B} : CC_k(\mathcal{A})((u)) \to \Omega_c^*(\Lambda\mathfrak{X}, L')((u)).$$

Proposition 4.6. $\tau_{\nabla,B} \circ (b + u\mathrm{B}) = (u\nabla' - 2\pi i u^2 \Omega) \circ \tau_{\nabla,B}$.

Proof. First, note that, according to Lemma 4.2, we have

$$\nabla^2 a = 2\pi i \partial B \wedge a = -2\pi i[B, a], \quad \forall a \in \Omega^*(\Gamma, L), \tag{13}$$

where $B \in \Omega^2(M, \mathbb{C})$ is considered as an element of $\Omega^2(\Gamma, L)$.

The rest of the proof is similar to [23, Proposition 5.4]. We will sketch the main steps below. Using Proposition 4.5(b), the relation $dB = \Omega$ and Eq. (13), we find that $(u\nabla' - 2\pi i u^2 \Omega) \circ \tau_{\nabla,B}(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_k)$ is the sum of the following two terms:

$$u \int_{\Delta^k} \mathrm{Tr}\big(\nabla(a_0) * e^{2\pi i u s_0 B} * \nabla(a_1) * \cdots * \nabla(a_k) * e^{2\pi i u s_k B}\big)\, ds_1 \cdots ds_k, \tag{14}$$

and

$$\sum_{i=1}^{k} (-1)^{i-1} u \int_{\Delta^k} \mathrm{Tr}\big(\tilde{a}_0 * e^{2\pi i u s_0 B} * \nabla(a_1) * \cdots * e^{2\pi i u s_{i-1} B} * [-2\pi i B, a_i] * e^{2\pi i u s_i B}$$
$$* \nabla(a_{i+1}) * \cdots * \nabla(a_k) * e^{2\pi i u s_k B}\big)\, ds_1 \cdots ds_k. \tag{15}$$

Using Proposition 4.5(a), one shows that the first term as given by Eq. (14) is equal to $(\tau_{\nabla,B} \circ u\mathrm{B})(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_k)$.

Using the formula

$$[a_i, e^K] = \int_0^1 e^{(1-s)K}[a_i, K]e^{sK}\, ds$$

(see [29]), one identifies the second term as given by Eq. (15) with $(\tau_{\nabla,B} \circ b)(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_k)$. □

As a consequence, the chain map $\tau_{\nabla,B}$ induces a homomorphism in cohomology

$$\tau_{\nabla,B} : HP_*(\mathcal{A}) \to H_c^*(\mathfrak{X}, \alpha).$$

In the next proposition, we study how $\tau_{\nabla,B}$ depends on the choice of ∇ and B. Assume that $\widetilde{\Gamma} \xrightarrow{\pi} \Gamma \rightrightarrows M$ admits connections θ_i, curvings B_i and 3-curvatures Ω_i, $i = 1, 2, 3$, as in Definition 3.3. Their corresponding covariant differentials are denoted by ∇_i, $i = 1, 2, 3$. Then, according to Lemma 3.1, $\theta_1 - \theta_0 = \partial A$ for some $A \in \Omega^1(M)$. Thus,

$$\nabla_1 - \nabla_0 = 2\pi i \partial A.$$

Let

$$\beta_{(\nabla_0, B_0), (\nabla_1, B_1)} = B_1 - B_0 - dA.$$

Lemma 4.7.

(a) $d\beta_{(\nabla_0,B_0),(\nabla_1,B_1)} = \Omega_1 - \Omega_0$;
(b) *the form* $\beta_{(\nabla_0,B_0),(\nabla_1,B_1)} \in \Omega^2(M)$ *is* Γ*-invariant, and its class in* $\Omega^2(\mathfrak{X})/d\Omega^1(\mathfrak{X})$ *is well defined*; *and*
(c) $\beta_{(\nabla_0,B_0),(\nabla_2,B_2)} = \beta_{(\nabla_0,B_0),(\nabla_1,B_1)} + \beta_{(\nabla_1,B_1),(\nabla_2,B_2)}$ *modulo* $d\Omega^1(\mathfrak{X})$.

Proof. (a) and (c) are obvious.

(b) Let $\beta = \beta_{(\nabla_0,B_0),(\nabla_1,B_1)}$. Then $\partial\beta = \partial(B_1 - B_0) - d\partial A = \partial(B_1 - B_0) - d(\theta_1 - \theta_0) = 0$. Hence β is Γ-invariant. Moreover, if $A' \in \Omega^1(M)$ is another one-form such that $\theta_1 - \theta_0 = \partial A'$, then $\beta' - \beta = d(A - A') \in d\Omega^1(M)^\Gamma$ since $\partial(A - A') = 0$. $\square$

Proposition 4.8. *Let* $u' = 2\pi i u$. *Consider the diagram*

$$\begin{tikzcd} HP_*(\mathcal{A}) \arrow[r, "\tau_{\nabla_0,B_0}"] \arrow[dr, "\tau_{\nabla_1,B_1}"'] & H^*\big(\Omega_c^*(\Lambda\mathfrak{X}, L')((u')), \nabla' - u'\Omega_0\big) \arrow[d, "e^{u'\beta_{(\nabla_0,B_0),(\nabla_1,B_1)}}"] \\ & H^*\big(\Omega_c^*(\Lambda\mathfrak{X}, L')((u')), \nabla' - u'\Omega_1\big). \end{tikzcd} \tag{16}$$

Then:

(a) *the vertical map is well defined, i.e. it is independent of the choice of* $\beta_{(\nabla_0,\Omega_0),(\nabla_1,\Omega_1)}$ *modulo* $d\Omega^1(\mathfrak{X})$; *and*
(b) *the diagram* (16) *is commutative.*

Proof. Let $\beta = \beta_{(\nabla_0,B_0),(\nabla_1,B_1)}$. First, note that $e^{u'\beta}$ is indeed a chain map, since $\forall\omega \in \Omega_c^*(S\Gamma, L')$, we have

$$(\nabla' - u'\Omega_1)\big(e^{u'\beta}\omega\big) = e^{u'\beta}(\nabla'\omega + u'd\beta\omega - u'\Omega_1\omega) = e^{u'\beta}(\nabla' - u'\Omega_0)\omega,$$

according to Lemma 4.7(a).

(a) It suffices to show that for all $A \in \Omega^1(M)^\Gamma$ and $\omega \in \Omega_c^*(\Lambda\mathfrak{X}, L')((u'))$, $e^{u'dA}\omega$ and ω are cohomologous in $H^*(\Omega_c^*(\Lambda\mathfrak{X}, L')((u')), \nabla' - u'\Omega_0)$.

Let $f(x) = \frac{e^x - 1}{x}$, and define

$$K : \Omega_c^*(\Lambda\mathfrak{X}, L')((u')) \to \Omega^{*-1}(\Lambda\mathfrak{X}, L')((u'))$$

by $K(\omega) = u'Af(u'dA)\omega$. Then

$$\begin{aligned} (\nabla' - u'\Omega_0)K(\omega) &= u'dAf(u'dA)\omega - u'Af(u'dA)\nabla'\omega - u'^2\Omega_0 Af(u'dA)\omega \\ &= \big(e^{u'dA} - 1\big)\omega - u'Af(u'dA)(\nabla'\omega - u'\Omega_0\omega) \\ &= e^{u'dA}\omega - \omega - K\big((\nabla' - u'\Omega_0)\omega\big). \end{aligned}$$

(b) Let $B_{1/2} = B_0 + dA$. It suffices to prove that the following diagrams are commutative:

$$\begin{tikzcd} & H^*\big(\Omega_c^*(\Lambda\mathfrak{X}, L')((u)), \nabla' - u'\Omega_0\big) \arrow[d, "\mathrm{Id}"] \\ HP_*(\mathcal{A}) \arrow[ur, "\tau_{\nabla_0, B_0}"] \arrow[r, "\tau_{\nabla_1, B_{1/2}}"'] \arrow[dr, "\tau_{\nabla_1, B_1}"'] & H^*\big(\Omega_c^*(\Lambda\mathfrak{X}, L')((u)), \nabla' - u'\Omega_0\big) \arrow[d, "e^{u'\beta}"] \\ & H^*\big(\Omega_c^*(\Lambda\mathfrak{X}, L')((u)), \nabla' - u'\Omega_1\big). \end{tikzcd} \tag{17}$$

For the lower triangle, note that

$$\begin{aligned} & e^{u'\beta} * \tilde{a}_0 * e^{u's_0 B_{1/2}} * \nabla_1 a_1 * \cdots * \nabla_1 a_k * e^{u's_k B_{1/2}} \\ &\quad = \tilde{a}_0 * e^{u's_0\beta} * e^{u's_0 B_{1/2}} * \nabla_1 a_1 * \cdots * \nabla_1 a_k * e^{u's_k\beta} * e^{u's_k B_{1/2}} \\ &\quad = \tilde{a}_0 * e^{u's_0 B_1} * \nabla_1 a_1 * \cdots * \nabla_1 a_k * e^{u's_k B_1}, \end{aligned}$$

where we use the relations $s_0 + \cdots + s_k = 1$ and $\beta = B_1 - B_{1/2}$. Note that since $\partial\beta = 0$, β commutes with every element in $\Omega^*(\Gamma, L)$ according to Lemma 4.2.

For the upper triangle, we proceed with the standard argument as in [23, Proposition 5.6] (but the proof is simpler here). Let $\Gamma^\sharp = \Gamma \times [0,1]$, and $\widetilde{\Gamma}^\sharp = \widetilde{\Gamma} \times [0,1]$. Then $\widetilde{\Gamma}^\sharp \to \Gamma^\sharp \rightrightarrows M \times [0,1]$ is an S^1-central extension. Let $\theta^\sharp = \theta + t\partial A$, which is considered as a 1-form on $\widetilde{\Gamma}^\sharp$. It is simple to see that $\theta^\sharp$ defines a connection of the S^1-central extension $\widetilde{\Gamma}^\sharp \to \Gamma^\sharp \rightrightarrows M \times [0,1]$. Moreover, since

$$d\theta^\sharp = d\theta + dt \wedge \partial A + t\, d\partial A = \partial(B + dt \wedge A + t\, dA),$$

and

$$d(B + dt \wedge A + t\, dA) = dB = \Omega,$$

it follows that $B^\sharp = B + dt \wedge A + t\, dA \in \Omega^2(M \times [0,1])$ is its curving, and Ω, being considered as an element in $\Omega^3(M \times [0,1])^{\Gamma^\sharp}$, is the 3-curvature. Applying Proposition 4.6, we have

$$\tau^\sharp \circ (b + u\mathrm{B}) = \big(u\nabla^{\sharp'} - 2\pi i u^2 \Omega\big) \circ \tau^\sharp, \tag{18}$$

where $\tau^\sharp = \tau_{\nabla^\sharp, B^\sharp}$. Note that when being restricted to $S\Gamma^\sharp (\cong S\Gamma \times [0,1])$, $\theta^\sharp$ is equal to θ since $\partial A|_{S\Gamma} = 0$. Therefore its corresponding covariant differential $\nabla^{\sharp'}$ is equal to $\nabla' + (dt)\frac{d}{dt}$, i.e. for any $v \in T_h S\Gamma$ and any t-dependent section $s_t \in \Gamma(L')$ being considered as a section of $L' \times [0,1] \to S\Gamma \times [0,1]$, we have

$$\nabla^{\sharp'}_{(v, \frac{d}{dt})} s_t = \nabla'_v s_t + \frac{ds_t}{dt}.$$

Contracting Eq. (18) by the vector field $\frac{d}{dt}$, dividing by u and integrating over [0, 1], we obtain

$$\tau_{\nabla_1,B_1} - \tau_{\nabla_0,B_0} = K \circ (b + u\mathrm{B}) - \left(u\nabla' - 2\pi i u^2 \Omega\right) \circ K,$$

where $K = u^{-1} \int_0^1 \iota_{\frac{d}{dt}} \tau^\sharp \, dt$. $\square$

4.4. Proof of the main theorem

Our goal in this subsection is to prove Theorem 1.1. The idea is that $K_\alpha^*(\mathfrak{X}) \otimes \mathbb{C}$, $HP_*(C_c^\infty(\Gamma, L))$ and $H^*(\mathfrak{X}, \alpha)$ agree locally, since an orbifold is locally a crossed-product of a manifold by a finite group; and each of these cohomology functors admits Mayer–Vietoris sequences, hence they agree globally.

For every open subset U of $|\mathfrak{X}|$, we denote by Γ_U the restriction of Γ to U, i.e.

$$\Gamma_U = \left\{\gamma \in \Gamma \mid s(\gamma), t(\gamma) \in \pi^{-1}(U)\right\},$$

where $\pi : M \to |\mathfrak{X}|$ is the projection. The corresponding orbifold of Γ_U is denoted by $\mathfrak{U}$.

Lemma 4.9. *For every open subset U of $|\mathfrak{X}|$, let $H^*(U, \alpha) = K_\alpha^*(\mathfrak{U}) \otimes \mathbb{C}$, $HP_*(C_c^\infty(\Gamma_U, L))$, or $H_c^*(\mathfrak{U}, \alpha)((u)) = H_c^{*+2\mathbb{Z}}(\mathfrak{U}, \alpha)$ $(* \in \mathbb{Z}/2\mathbb{Z})$.*

(a) *If (U_i) is an increasing net of open subsets of $|\mathfrak{X}|$ which covers $|\mathfrak{X}|$, then $H^*(|\mathfrak{X}|, \alpha) = \lim_i H^*(U_i, \alpha)$;*
(b) *If $|\mathfrak{X}|$ is covered by two open subsets U and V, then there is a Mayer–Vietoris exact sequence*

$$\begin{tikzcd}
H^0(U \cap V, \alpha) \arrow[r] & H^0(U, \alpha) \oplus H^0(V, \alpha) \arrow[r] & H^0\left(|\mathfrak{X}|, \alpha\right) \arrow[d] \\
H^1\left(|\mathfrak{X}|, \alpha\right) \arrow[u] & H^1(U, \alpha) \oplus H^1(V, \alpha) \arrow[l] & H^1(U \cap V, \alpha). \arrow[l]
\end{tikzcd}$$

Proof. (a) For K_*, this follows from the facts that K-theory commutes with inductive limits, and that $(C^*(\Gamma_{U_i}, L))$ is an increasing net of ideals in $C^*(\Gamma, L)$ whose union is dense in $C^*(\Gamma, L)$.

For HP_* and $H_c^*(-, \alpha)$, this is obvious.

(b) For K_*, this is proven in [32, Proposition 3.9]. For HP_*, this follows from [14] (see also [27]) and for $H_c^*(-, \alpha)$, the proof is standard using smooth partitions of unity (see Lemma 4.10 below). $\square$

Lemma 4.10. *Given an open cover $\mathcal{U} = (U_i)$ of $|\mathfrak{X}|$, there exists a partition of unity subordinate to $\mathcal{U}$ consisting of smooth functions on $\mathfrak{X}$.*

Proof. Let $\pi : M \to |\mathfrak{X}|$ be the projection map. For each i, choose an open submanifold V_i of M such that $\pi(V_i) = U_i$. Let $\Gamma' \rightrightarrows \coprod V_i$ be the pull-back of $\Gamma \rightrightarrows M$ via the étale map $\coprod V_i \to M$. Let $c = (c_i)_{i \in I}$ be a smooth cutoff function for the proper groupoid $\Gamma' \rightrightarrows \coprod V_i$

[31, Proposition 6.7]. By definition, $c_i : V_i \to \mathbb{R}_+$ is a smooth function such that for all i and all $x \in V_i$, $\sum_j \sum_{g \in \Gamma^x} c_j(s(g)) = 1$ (by convention, c_j is extended by zero outside V_j). Let

$$\varphi_i(x) = \sum_{g \in \Gamma^x} c_i\big(s(g)\big) \quad (x \in M).$$

Then φ_i is clearly Γ-invariant, smooth, and $\sum \varphi_i = 1$. $\square$

Thus, in order to prove Theorem 1.1, it suffices, by induction using a five-lemma argument and passing to the inductive limit, to prove the following

Proposition 4.11. *For every $\bar{x} \in |\mathfrak{X}|$, there exists an open neighborhood U such that for every open subset $V \subset U$, the homomorphisms*

$$K_*\big(C_c^\infty(\Gamma_V, L)\big) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}} HP_*\big(C_c^\infty(\Gamma_V, L)\big) \xrightarrow{\tau} H_c^*(\mathfrak{V}, \alpha) \tag{19}$$

are isomorphisms.

Lemma 4.12. *For each $\bar{x} \in |\mathfrak{X}|$, there exists an open neighborhood U of $\bar{x}$ such that $\alpha_{|U} \in H^3(\mathfrak{X}|_U, \mathbb{Z})$ is represented by an S^1-central extension*:

$$S^1 \to U' \rtimes \widetilde{G} \to U' \rtimes G \rightrightarrows U', \tag{20}$$

which is the pull back of a group S^1-central extension of finite order:

$$S^1 \to \widetilde{G} \to G, \tag{21}$$

induced by a $\mathbb{Z}_n$-central extension:

$$1 \to \exp\left(\frac{2\pi i \mathbb{Z}}{n}\right) \to G' \to G \to 1, \quad \text{where } \widetilde{G} = G' \times_{\exp(\frac{2\pi i \mathbb{Z}}{n})} S^1. \tag{22}$$

Here U' is an Euclidean ball and G is the stabilizer of $\bar{x}$ acting on U' by isometries.

Proof. First take a nice orbifold chart around $\bar{x}$ of the form $U' \rtimes G$ as in [25], where G is a finite group. Since U' is G-contractible, we have

$$H^3_G(U', \mathbb{Z}) \cong H^3_G(pt, \mathbb{Z}).$$

It thus follows that the class $\alpha_{|U}$ is represented by an S^1-central extension of $U' \rtimes G$, which is the pull back from an S^1-central extension of the form (21). Note that the class $\alpha' \in H^3(G_\bullet, \mathbb{Z})$ of the extension (21) must be torsion since the image of α' in $H^3(G_\bullet, \mathbb{R}) = H^3(G_\bullet, \mathbb{Z}) \otimes \mathbb{R}$ is zero.

It follows that, if n is such that $n\alpha' = 0$, the extension (21) is obtained from (22). $\square$

Lemma 4.13. *Assume that Γ is an étale proper groupoid, $S^1 \to \widetilde{\Gamma} \to \Gamma \rightrightarrows M$ is a torsion class S^1-central extension induced from a $\mathbb{Z}_n$-central extension $\exp \frac{2\pi i \mathbb{Z}}{n} \to \widetilde{\Gamma}' \to \Gamma \rightrightarrows M$. Then there is an induced flat gerbe connection on $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$.*

Proof. Let $L = \widetilde{\Gamma} \times_{S^1} \mathbb{C}$ be the induced line bundle over Γ. For each $q \geqslant 0$, the space $\Omega^q(\widetilde{\Gamma}')$ of q-forms on $\widetilde{\Gamma}'$ admits a decomposition

$$\Omega^q(\widetilde{\Gamma}') = \bigoplus_{k=0}^{n-1} \Omega^q\big(\Gamma, L^{\otimes k}\big), \tag{23}$$

where $\Omega^q(\Gamma, L^{\otimes k})$ can be naturally identified with the space $\{\omega \in \Omega^q(\Gamma') \mid z\cdot\omega = (\exp \frac{2\pi i z k}{n})\omega,\ \forall z \in \mathbb{Z}_n\}$ (this is analogue to [32, Proposition 3.2]), Hence $z\cdot$ denotes the induced $\mathbb{Z}_n$-action on $\Omega^q(\widetilde{\Gamma}')$. Indeed, if we consider the operator T on $\Omega^q(\widetilde{\Gamma}')$ given by $T(\omega) = e^{-\frac{2\pi i}{n}} \cdot \omega$, then Eq. (23) is just the decomposition of $\Omega^q(\widetilde{\Gamma}')$ into the eigenspaces of T:

$$\Omega^q(\widetilde{\Gamma}') = \bigoplus_{k=0}^{n-1} \ker\big(T - e^{\frac{2\pi i k}{n}} \mathrm{Id}\big).$$

It is simple to see that each eigenspace of T, i.e. $\Omega^q(\Gamma, L^{\otimes k})$ is stable under the de Rham differential $d : \Omega^*(\widetilde{\Gamma}') \to \Omega^{*+1}(\widetilde{\Gamma}')$. In particular, one obtains a degree 1 differential operator ∇_1 on $\bigoplus_q \Omega^q(\Gamma, L)$, which is easily seen to satisfy the axioms of a covariant differential. Therefore there is an induced linear connection on $L \to \Gamma$, which must be flat since $d^2 = 0$. To show that ∇_1 satisfies the gerbe connection condition, we have to check that $\nabla_1(f_1 * f_2) = \nabla_1 f_1 * f_2 + f_1 * \nabla_1 f_2$ (see Lemma 4.3), where $*$ is the convolution product on $C_c^\infty(\Gamma, L)$. Let $*'$ be the convolution product on $\widetilde{\Gamma}'$. Then, identifying $C_c^\infty(\Gamma, L)$ with a subspace of $C_c^\infty(\widetilde{\Gamma}')$ as above, we have $f_1 *' f_2 = n f_1 * f_2$. So it suffices to show that $d(f_1 *' f_2) = df_1 *' f_2 + f_1 *' df_2$. The latter follows from Lemma 4.3 applied to the groupoid $\widetilde{\Gamma}'$ endowed with the trivial gerbe. $\square$

Note that the above construction works for compactly supported differential forms as well. In particular,

$$\mathcal{A}' = \bigoplus_{k=0}^{n-1} C_c^\infty\big(\Gamma, L^{\otimes k}\big). \tag{24}$$

We thus obtain the following decompositions:

$$K_*\big(C_c^\infty(\widetilde{\Gamma}')\big) = \bigoplus_{k=0}^{n-1} K_*\big(C_c^\infty\big(\Gamma, L^{\otimes k}\big)\big),$$

$$HP_*\big(C_c^\infty\big(\widetilde{\Gamma}'\big)\big) = \bigoplus_{k=0}^{n-1} HP_*\big(C_c^\infty\big(\Gamma, L^{\otimes k}\big)\big).$$

On the other hand, Lemma 4.13 and its proof imply that

$$H^*\big(\Omega_c^*(S\widetilde{\Gamma}')^{\widetilde{\Gamma}'}((u))\big) = \bigoplus_{k=0}^{n-1} H^*\big(\Omega_c^*\big(S\Gamma, L^{'\otimes k}\big)^{\Gamma}((u)), \nabla_1^{'k}\big), \tag{25}$$

where ∇_1 is the connection on $L \to \Gamma$ as in Lemma 4.13, and ∇_1^k its induced connection on $L^{\otimes k} \to \Gamma$ while the upscript $'$ stands for their restrictions to the closed loops $S\Gamma$. We are now ready to prove Proposition 4.11.

Proof of Proposition 4.11. Assume that we already have established the following isomorphisms:

$$K_*\big(C_c^\infty(\widetilde{\Gamma}')\big) \otimes \mathbb{C} \xrightarrow{\text{ch}} HP_*\big(C_c^\infty(\widetilde{\Gamma}')\big) \xrightarrow{\tau} H^*\big(\Omega_c^*(S\widetilde{\Gamma}')^{\widetilde{\Gamma}'}((u))\big), \tag{26}$$

where τ is the homomorphism constructed as in Eq. (12) using the trivial connection and curving 0 by considering the groupoid $\widetilde{\Gamma}'$ being equipped with the trivial gerbe. More precisely,

$$\tau(\tilde{a}_0 \otimes a_1 \otimes \cdots \otimes a_k) = \frac{1}{k!}\operatorname{Tr}(\tilde{a}_0 * da_1 * \cdots * da_k). \tag{27}$$

Then the isomorphisms (26) induce isomorphisms on eigenspaces of T (since all maps in (26) commute with T). Thus

$$K_*\big(C_c^\infty(\Gamma, L)\big) \otimes \mathbb{C} \xrightarrow{\text{ch}} HP_*\big(C_c^\infty(\Gamma, L)\big) \xrightarrow{\tau_{\nabla_1,0}} H^*\big(\Omega_c^*\big((S\Gamma, L')^{\Gamma}((u)), \nabla_1'\big)\big)$$

are isomorphisms. From Proposition 4.8, it follows that

$$K_*\big(C_c^\infty(\Gamma, L)\big) \otimes \mathbb{C} \xrightarrow{\text{ch}} HP_*\big(C_c^\infty(\Gamma, L)\big) \xrightarrow{\tau_{\nabla,B}} H^*\big(\Omega_c^*(S\Gamma, L')^{\Gamma}((u)), \nabla_1' - 2\pi i u\Omega\big)$$

are isomorphisms.

It thus remains to prove that (26) are indeed isomorphisms. Therefore we are reduced to the case when Γ is Morita equivalent to the crossed-product $U \rtimes G$ of a manifold by a finite group and $\alpha = 0$. Of course, we may also assume that $V = U$, since $\Gamma_{|V}$ satisfies the same properties. $\square$

In this case, using the two lemmas below, we can replace $\Gamma \rightrightarrows M$ by its Morita equivalent groupoid $U \rtimes G \rightrightarrows U$.

Lemma 4.14. *Assume that $\Gamma \rightrightarrows M$ is an étale proper groupoid. Let (U_i) be an open cover of M, $M' = \coprod U_i$ and $\Gamma' = \{(i, g, j) \mid g \in \Gamma_{U_j}^{U_i}\}$ the pull back of Γ under the étale map $\coprod U_i \to M$. The following diagram is commutative*:

$$\begin{array}{ccc} HP_*\big(C_c^\infty(\Gamma')\big) & \xrightarrow{\quad\phi\quad} & HP_*\big(C_c^\infty(\Gamma)\big) \\ \Big\downarrow{\scriptstyle\tau'} & & \Big\downarrow{\scriptstyle\tau} \\ H^*\big(\Omega_c^*(S\Gamma')^{\Gamma'}((u))\big) & \xrightarrow{\quad\sim\quad} & H^*\big(\Omega_c^*(S\Gamma)^{\Gamma}((u))\big) \end{array} \tag{28}$$

where τ and τ' are defined by Eq. (27)*, and ϕ is given by*

$$\phi(\tilde{a}_0 \otimes \cdots \otimes a_k)(g_0, \ldots, g_k) = \sum_{i_0,\ldots,i_k} \tilde{a}_0(i_0, g_0, i_1) \otimes \cdots \otimes a_n(i_k, g_k, i_0)$$

(*note that $g \mapsto a_j(i_j, g, i_{j+1})$ is a smooth compactly supported function on $\Gamma_{U_{i_{j+1}}}^{U_{i_j}}$. Therefore it* *can be considered as a smooth compactly supported function on Γ). Moreover, ϕ is an isomorphism.*

Proof. The fact that ϕ is an isomorphism, i.e. that $HP_*(C_c^\infty(\Gamma))$ only depends on the Morita equivalence class of the étale proper groupoid $\Gamma \rightrightarrows M$ is standard; e.g. see [11,13].

That the diagram (28) commutes follows from a direct and elementary computation: for any $h \in S\Gamma$, $(\tau \circ \phi)(\tilde{a}_0 \otimes \cdots \otimes a_k)(h)$ and $\tau'(\tilde{a}_0 \otimes \cdots \otimes a_k)(h)$ are both equal to

$$\frac{1}{k!} \sum_{g \in \Gamma^{s(h)}} \sum_{g_0 \cdots g_k = g^{-1}hg} \sum_{i_0,\dots,i_k} \tilde{a}_0(i_0, g_0, i_1)\, da_1(i_1, g_1, i_2) \cdots da_k(i_k, g_k, i_0). \qquad \square$$

Lemma 4.15. *Let Γ and Γ' be as in Lemma* 4.14. *The following diagram commutes, and the horizontal maps are isomorphisms*:

$$\begin{array}{ccc} K_*\big(C_c^\infty(\Gamma')\big) & \longrightarrow & K_*\big(C_c^\infty(\Gamma)\big) \\ \big\downarrow \mathrm{ch} & & \big\downarrow \mathrm{ch} \\ HP_*\big(C_c^\infty(\Gamma')\big) & \longrightarrow & HP_*\big(C_c^\infty(\Gamma)\big). \end{array}$$

Proof. Let $\Gamma'' = \Gamma \times (I \times I)$, where I is the index set of (U_i), and $I \times I$ is the pair groupoid equipped with the product $(i, j)(j, k) = (i, k)$. The inclusion of Γ' as an open subgroupoid of Γ'' induces a $*$-homomorphism $C_c^\infty(\Gamma') \to C_c^\infty(\Gamma'')$. Choosing $i_0 \in I$, the inclusion $g \mapsto (g, i_0, i_0)$ also induces a $*$-homomorphism $C_c^\infty(\Gamma) \to C_c^\infty(\Gamma'')$. Moreover, these $*$-homomorphisms are Morita equivalences, hence the following diagram

$$\begin{array}{ccccc} K_*\big(C_c^\infty(\Gamma')\big) & \longrightarrow & K_*\big(C_c^\infty(\Gamma'')\big) & \longleftarrow & K_*\big(C_c^\infty(\Gamma)\big) \\ \big\downarrow \mathrm{ch} & & \big\downarrow \mathrm{ch} & & \big\downarrow \mathrm{ch} \\ HP_*\big(C_c^\infty(\Gamma')\big) & \longrightarrow & HP_*\big(C_c^\infty(\Gamma'')\big) & \longleftarrow & HP_*\big(C_c^\infty(\Gamma)\big) \end{array}$$

is commutative and the horizontal arrows are isomorphisms. $\square$

Now, the case $\Gamma = U \rtimes G$ is covered by Baum and Connes [3, Theorem 1.19]. More precisely, the proof of Proposition 4.11 and Theorem 1.1 will be completed thanks to the following:

Lemma 4.16. *Let G be a finite group acting on a manifold M. The following homomorphisms*

$$K_*\big(C_c^\infty(M \rtimes G)\big) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}} HP_*\big(C_c^\infty(M \rtimes G)\big) \xrightarrow{\tau} H^*\big(\Omega_c^*(\hat{M})^G((u))\big) \tag{29}$$

are isomorphisms, where $\hat{M} = \coprod_{g \in G} M^g$, Ω_c^* denotes the space of differential forms which are compactly supported in $\hat{M}/G$, and τ is the map defined by Eq. (27). *Moreover, $\tau \circ \mathrm{ch}$ is the Baum–Connes' Chern character* [3] *and hence is an isomorphism.*

Proof. This is a well-known result. However since we cannot locate it in literature, we sketch a proof below.

Let us treat some special cases first. If G is the trivial group then $\tau \circ \mathrm{ch}$ is the usual Chern character and τ is Connes–Hochschild–Kostant–Rosenberg's isomorphism [10,28].

If M is a point, the maps (29) become

$$R(G) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}} \mathbb{C}[G]^G \xrightarrow{\tau} \mathbb{C}[G]^G,$$

where $\mathbb{C}[G]^G$ denotes the complex-valued functions on G which are invariant under conjugation. Then ch is the character map; we will show that τ is the identity map. For this purpose, let us check that $\tau \circ \mathrm{ch}$ is also the character map. Let π be an irreducible representation of G, χ its character and d_π its dimension. Let

$$f(g) = \frac{d_\pi \chi(g)}{\#G}.$$

Then the corresponding element $P \in \mathbb{C}[G] = C^*(G)$ is a projection which corresponds to the element $d_\pi[\pi] \in R(G)$ [15] (the term $\#G$ comes from the fact that we use the counting measure instead of the normalized Haar measure). Moreover, one checks immediately that

$$(\tau \circ \mathrm{ch})\big(d_\pi[\pi]\big)(h) = \big(\tau \circ \mathrm{ch}\big([P]\big)\big)(h) = \big(\tau(f)\big)(h) = \sum_g f\big(g^{-1}hg\big) = d_\pi \chi(h).$$

Now the case when G acts trivially on M follows from the isomorphisms $K_*(C_c^\infty(M) \rtimes G) = K_*(C_c^\infty(M)) \otimes R(G)$ and the analogue isomorphisms for HP_* and H_c^*.

Let us turn to the general case. Let $\sigma(M,G) = \max\{\#\mathrm{stab}(x) \mid x \in M\}$. We proceed by induction on $\sigma(M,G)$. If $\sigma(M,G) = 1$, then G acts freely on M. After replacing $M \rtimes G$ by the Morita equivalent manifold M/G we are reduced to Connes–Hochschild–Kostant–Rosenberg's theorem as above.

Suppose the proposition is proven for $\sigma(M,G) < N$ and assume $\sigma(M,G) = N$. Let $U = \{x \in M \mid \#\mathrm{stab}(x) < N\}$ and $F = M - U$. Then U is an open invariant subset of M such that $\sigma(U,G) < N$ and F is a closed submanifold of M. By the induction assumption, the proposition is true for U, so by using six-term exact sequences associated to the pair (U,F), we just have to show the proposition for the space F. I.e. we are reduced to the case when all stabilizers have cardinality N. Now, let $M^H = \{x \in M \mid \mathrm{stab}(x) = H\}$ and let M_H be the saturation of M^H. Choose a representative H_i for each conjugacy class of subgroups of cardinal N. Since the stabilizer of a point in M is conjugate to one of the H_i's, $M = \coprod M_{H_i}$ is a finite partition of M. Moreover, since all stabilizers have cardinality N, we have $M^H = \{x \in M \mid \mathrm{stab}(x) \supseteq H\}$. Hence M^H is closed for all H. It follows that $M = \coprod M_{H_i}$ is a partition of M into finitely many open and closed subsets, so we can assume that $M = M_H$ for some H. Then since

$$M^H \cdot g = M^{g^{-1}Hg},$$

we see that $M \rtimes G$ is Morita equivalent to $M^H \rtimes K$, where K is the normalizer of H. Hence we can assume that, after replacing M by M^H and G by K, H is normal and $\mathrm{stab}(x) = H$ for all x, i.e. the action of G comes from a free action of G/H.

Now, for all $x \in M/G = M/(G/H)$, there exists a neighborhood V of x such that $\pi^{-1}(V)$ is equivariantly diffeomorphic to $V \times (G/H)$. Cover M/G by such open subsets V_i. Using Mayer–Vietoris exact sequences and an induction argument, we can assume that $M = V \times (G/H)$. Therefore $M \rtimes G$ is Morita equivalent to the crossed-product $V \times H$ of a manifold by a trivial group action, which is the case considered earlier. $\square$

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